[This question paper contains 4 printed pages.]

Sr. No. of Question Paper : 1161 **E** **Your Roll No...............**

Unique Paper Code : 235201

Name of the Course : **B.Sc. (Hons.) Mathematics**

Name of the Paper : Differential Equations and Mathematical Modelling – I

Semester : II

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **all** questions.
3. Use of Scientific Calculators is allowed.

SECTION I

1. Attempt any **three** of the following : (5+5+5)

(a) Find general solution of the differential equation

$$\frac{dy}{dx} = 1 + x + y + xy$$

(b) Solve the differential equation

$$\frac{dy}{dx} = \sqrt{x + y + 1}$$

(c) Solve the differential equation

$$x\frac{d^2y}{dx^2} = \frac{dy}{dx}$$

(d) Solve the differential equation by first finding an integrating factor

$$[y^2(x + 1) + y]\,dx + (2xy + 1)\,dy = 0$$

2. Attempt any **two** of the following : (5+5)

(a) A spherical tank of radius 4 ft is full of gasoline when a circular bottom hole with radius one inch is opened. How long will be required for all the gasoline to drain from the tank ?

(b) Assume that a body moving with velocity v encounters resistance of the form $\frac{dv}{dt} = -k\,v^{3/2}$. Show that $v(t) = \frac{4vo}{\left(kt\sqrt{vo}+2\right)^2}$ and that

$$x(t) = x_0 + \frac{2}{k}\sqrt{vo}\left(1 - \frac{2}{kt\sqrt{vo}+2}\right).$$

(c) A cake is removed from an oven at 210°F and left to cool at room temperature, which is 70°F. After 30 minutes the temperature of the cake is 140°F. When will it be 100°F ?

SECTION II

3. Attempt any **two** of the following : (8+8)

(a) A rabbit population consisting of 1000 rabbits has a per capita birth rate of 10 rabbits/month and a per capita death rate of 2 rabbits/month. Also 40 traps are set each week and they are always filled. Assuming that each trap can contain 1 rabbit, find the rabbit population at any time t.

(b) A pill is taken for common cold. The pill dissolves in the gastrointestinal tract and ingredients of pill are diffused in the bloodstream. They are carried to the locations in which they act and are removed from the blood by kidney and liver. Assume that x(t) is the level of drug in the gastrointestinal tract at time t, y(t) is the level of drug in the bloodstream at time t. Examine the change in the level of drug in the gastrointestinal tract and bloodstream. Assume $x(0) = x_0$, $y(0) = 0$.

(c) Obtain an expression for half-life of a radioactive substance.

SECTION III

4. Attempt any **three** of the following : (6+6+6)

(a) Solve the differential equation :

$$x^2y'' + 2xy' - 12y = 0$$

for $x > 0$.

(b) Show first that the three solution $y_1(x) = e^x$, $y_2(x) = xe^x$, and $y_3(x) = x^2e^x$ of the third-order differential equation $y^{(3)} - 3y'' + 3y' - y = 0$ are linearly independent on the open interval $x > 0$. Then find a particular solution of the above differential equation that satisfies the initial condition $y(0) = 2$, $y'(0) = 0$, $y''(0) = 0$.

(c) Use the method of undetermined coefficients to find the particular solution of the differential equation :

$$2y'' + 4y' + 7y = x^2.$$

(d) Use the method of variation of parameters to find a particular solution of the differential equation

$$y'' - 4y' + 4y = 2e^{2x}.$$

(e) A body with mass 250 g is attached to the end of a spring that is stretched 25 cm by a force of 9 N. At time t = 0 the body is pulled 1 m to the right, stretching the spring, and the set in motion with an initial velocity of 5 m/s to the left. Find x(t) in the form $C\cos(\omega_0 t - \alpha)$ and find the amplitude of the motion of the body.

SECTION IV

5. Attempt any **one** of the following : (16)

(a) Consider a disease where all those who are infected remain contagious for life. Ignore births and deaths. Write down suitable word equations for rate of change in Susceptible and Infective and hence develop a pair of differential equations. Sketch the phase plane for this model.

(b) Develop a model with two differential equations describing a predator-prey interaction. What assumptions have been made for this model ? Check the model in the limiting cases of (i) prey with no predators (ii) predators with no prey. Sketch the phase plane for this model.

[This question paper contains 4 printed pages.]

Sr. No. of Question Paper : 1162 **E** **Your Roll No...............**

Unique Paper Code : 235203

Name of the Course : **B.Sc. (Hons.) Mathematics**

Name of the Paper : Analysis – II [MAHT–202]

Semester : II

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **three** parts from each question.
3. **All** questions are compulsory.

1. (a) Use the $\in - \delta$ definition of the limit to show that

$$\lim_{x \to 2} \frac{x^3 - 4}{x^2 + 1} = \frac{4}{5}.$$

(b) Suppose that the functions f and g have limits in R as $x \to \infty$ and that $f(x) \le g(x)$ for all $x \in (a, \infty)$. Prove that $\lim_{x \to \infty} f \le \lim_{x \to \infty} g$.

(c) State squeeze theorem and hence prove

$$\lim_{x \to 0} \left(\frac{\cos x - 1}{x} \right) = 0.$$

P.T.O.

(d) Use sequential criterion to show that the following limit does not exist in R

$$\lim_{x\to 0}\left(\sqrt{\frac{1}{x}}\right), x > 0 \qquad (5,5,5,5)$$

2. (a) Show that $\lim_{x\to 0}\left(\frac{1}{1+e^{\frac{1}{x}}}\right)$, $x \neq 0$ does not exist in R.

(b) Let $A \subseteq R$, let f: $A \to R$ and let $|f|$ be defined by $|f|(x) = |f(x)|$ for $x \in A$. Show that if f is continuous on A, then $|f|$ is continuous on A. Is the converse true ?

(c) Suppose that f: $R \to R$ is continuous on R and that $f(r') = 0$ for every irrational number r'. Prove that $f(x) = 0$ for all $x \in R$.

(d) Let f: $R \to R$ be continuous on R and let $P = \{x \in R : f(x) > 0)$. If $c \in P$, show that there exists a neighborhood $V_\delta(c) \subseteq P$. (5,5,5,5)

3. (a) (i) Show that $x2^x = 1$ for some x in (0, 1).

(ii) Give an example to show that the converse of Intermediate value theorem may not hold.

(b) Let $I = [a, b]$ be an interval and let f: $I \to R$ be continuous on I. If $f(a) < 0 < f(b)$, then prove that there exists a point $c \in I$ such that $f(c) = 0$.

(c) Prove that if the two functions f and g are uniformly continuous on R, then the composite function fog is uniformly continuous on R.

(d) Define uniform continuity of a function f defined on R. Show that $f(x) = x^3$ $x \in [a, b]$ is uniformly continuous on the given domain.

(5,5,5,5)

4. (a) Define differentiability of a function f at $c \in I$, where $f: I \to R$ and I is an interval in R. Determine the set of all points of R on which $f(x) = e^{|x|}$, $x \in R$ is not differentiable. Justify.

(b) Suppose that $f: R \to R$ is differentiable at c and that $f(c) = 0$. Show that $g(x) = |f(x)|$ is differentiable at c if and only if $f'(c) = 0$.

(c) Let $f: I \to R$ be differentiable on the interval I. Prove that f is decreasing on I if and only if $f'(x) \leq 0$ for all $x \in I$.

(d) If f is differentiable on $I = [a, b]$, and if 'k' is a number between $f'(a)$ and $f'(b)$, then prove that there is at least one point 'c' in (a,b) such that $f'(c) = k$. (5,5,5,5)

5. (a) Let f be differentiable on R and that $f(0) = 0$, $f(1) = 1$ and $f(2) = 1$. Show that $f'(x) = \frac{1}{2}$, for some $x \in (0,2)$. State the theorems used.

(b) State Taylor's Theorem and use it to show that :

$$x - \frac{x^2}{2} + \ldots\ldots - \frac{1}{2k}x^{2k} < \log(1+x) < x - \frac{x^2}{2} + \ldots\ldots + \frac{1}{2k+1}x^{2k+1},$$ for any

$k \in N$, and for all $x > 0$.

(c) Obtain Maclaurin's series expansion for the function $f(x) = e^{2x}$, $x \in R$.

P.T.O.

(d) Define a convex function on an interval in R. Check the convexity for the function $f(x) = (1 + x)^{1/3}$, $x \in (-1, 0)$, on the given interval. State the theorem used. (5,5,5,5)

[This question paper contains 4 printed pages.]

Sr. No. of Question Paper : 1163 **E** **Your Roll No...............**

Unique Paper Code : 235204

Name of the Course : **B.Sc. (Hons.) Mathematics**

Name of the Paper : Probability & Statistics – MAHT 203

Semester : II

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. In all there are **six** questions.
3. Question No. **1** is compulsory and it contains **five** parts of **3** marks each.
4. In Question No. **2** to **6,** attempt any **two** parts from **three** parts. Each part carries **6** marks.
5. Use of scientific calculator is allowed.

1. (i) If C_1 and C_2 are events in a sample space S. Then prove that

$$P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2).$$

(ii) A bowl contains 16 chips, of which 6 are red, 7 are white and 3 are blue. If four chips are taken at random and without replacement, find the probability that : (a) each of the 4 chips is red; (b) none of the 4 chips is red; (c) there is at least 1 chip of each color.

(iii) Let X_1 and X_2 have the joint pdf $f(x_1,x_2) = x_1 + x_2$, $0 < x_1 < x_2 < 1$, zero elsewhere. Find the marginal pdfs of X_1 and X_2.

(iv) Show that if a random variable has a uniform density with the parameters α and β, the probability that it will take on a value less than $\alpha + p(\beta - \alpha)$ is equal to p.

(v) If X is a random variable with $E(X) = 3$ and $E(X^2) = 13$. Use the Chebyshev inequality to determine a lower bound for $P(-2 < X < 8)$.

2. (i) Let $\{C_n\}$ be an arbitrary sequence of events. Then prove that

$$P\left(\bigcup_{n=1}^{\infty} C_n\right) \leq \sum_{n=1}^{\infty} P(C_n).$$

(ii) If X is a random variable having the Standard Normal distribution and $Y = X^2$, show that $Cov(X, Y) = 0$ even though X and Y are evidently not independent.

(iii) Find the moment generating function of the gamma distribution and hence find the mean and the variance.

3. (i) State Negative Binomial distribution and find its mean and variance.

(ii) Show that if X is a random variable having the Poisson distribution with parameter λ and $\lambda \to \infty$, then the moment generating function of $Z = \frac{x - \lambda}{\sqrt{\lambda}}$, that is, that of a standardized Poisson random variable, approaches the moment generating function of the standard normal distribution.

(iii) Find the mean and the variance of the exponential distribution.

4. (i) (a) If the probability is 0.75 that an applicant for a driver's license will pass the road test on any given try, what is the probability that an applicant will finally pass the test on the fourth try ?

(b) If X has an exponential distribution, show that

$$P(X \geq t + T/X \geq T) = P(X \geq t).$$

(ii) Given the joint density $f(x, y) = 6x$, $0 < x < y < 1$, zero elsewhere. Find $\mu_{Y/X}$.

(iii) If (X, Y) has a bivariate normal distribution, find the marginal distribution of X and Y. Under what conditions X and Y are independent ?

5. (i) If the regression of Y on X is linear, then show that

$$\mu_{Y/X} = \mu_2 + \frac{\rho\sigma_2}{\sigma_1}(x - \mu_1).$$

(ii) (a) Let X be a random variable such that $P(X \leq 0) = 0$ and let $\mu = E(X)$ exists. Show that $P(X \geq 2\mu) \leq \frac{1}{2}$.

(b) Find the cdf of a random variable X having pdf $f(x) = 6x(1 - x)$, $0 < x < 1$, zero elsewhere.

(iii) Suppose X_1 and X_2 are random variables of the discrete type which have the joint pmf

$$p(x_1, x_2) = \frac{x_1 + 2x_2}{18}, (x_1, x_2) = (1,1), (1,2), (2,1), (2,2), \text{ zero elsewhere.}$$

Determine the mean and the variance of X_2 given $X_1 = x_1$ for $x_1 = 1$ or 2. Also compute $E(3X_1 - 2X_2)$.

6. (i) State and Prove Chapman-Kolmogorov's equations.

(ii) Suppose that the chance of rain tomorrow depends on previous weather conditions only through whether or not it is raining today and not on past weather conditions. Suppose also that if it rains today, then it will rain tomorrow with the probability 0.7; and if it does not rain today then it will rain tomorrow with the probability 0.4. Express this model as a Markov chain and find its transition probability matrix. Calculate the probability that it will rain four days from today given that it is raining today.

(iii) State and prove Chebyshev's Theorem.

[This question paper contains 2 printed pages.]

Sr. No. of Question Paper : 1289 **E** **Your Roll No................**

Unique Paper Code : 222281

Name of the Course : **B.Sc. (Hons.) Mathematics/B.Sc. Mathematical Science**

Name of the Paper : Physics – I

Semester : II

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **five** questions in all including Question No. **1** which is compulsory.

1. Answer any **five** of the following questions:

(a) Determine whether the force $\vec{F} = 3xy\hat{i} - y\hat{j}$ is conservative or not.

(b) Prove: $\vec{A}\times(\vec{B}\times\vec{C}) + \vec{B}\times(\vec{C}\times\vec{A}) + \vec{C}\times(\vec{A}\times\vec{B}) = 0$.

(c) State and prove the law of conservation of angular momentum.

(d) If the plane of vibration of the incident beam makes an angle of 30° with the optic axis, compare the intensities of extraordinary and ordinary light.

(e) Give the physical interpretation of

(i) Divergence of a vector field and

(ii) Curl of a vector field.

(f) Define scalar and vector fields. Give examples.

(g) Explain how light can be polarized using reflection method? (3×5=15)

2. (a) State and prove work-energy theorem. (7)

(b) What are conservative and non-conservative forces? Give an example for each. (4)

(c) Define the terms torque and angular momentum. Derive a mathematical relation between them. (4)

3. (a) Derive an expression for the moment of inertia of a solid sphere about an axis passing through its centre. (8)

(b) What is centre of mass? Show that in the absence of external forces the velocity of the centre of mass remains constant. (7)

4. (a) What are forced oscillations? Write a differential equation for a forced oscillator and discuss the significance of each of the terms appearing in the equation. (7)

(b) A simple harmonic motion has amplitude of 5 cm and a frequency of 12 Hz. At time t = 0 the particle has a displacement equal to its amplitude. Write an expression for the displacement, velocity and acceleration of the particle. Also plot the displacement versus time curve. (8)

5. (a) Explain the phenomenon of interference in a thin film. Show that the intensity distribution pattern for reflected and transmitted light are complementary to each other. (10)

(b) A thin film of plastic of refractive index 1.45 for light of wavelength 589 nm is inserted normally in path of one of the interfering beams. The central bright band shifts through 10 fringes. Find the thickness of the film. (5)

6. (a) What is Rayleigh's criterion of resolution? Find the expression for the resolving power of a telescope. (5)

(b) What is a plane transmission grating? Discuss its theory. How it can be used to determine the wavelength of light? (10)

7. (a) What is polarization of light? Discuss the production and analysis of circularly and elliptically polarized light. 10

(b) Calculate the thickness of a half wave plate of quartz when a light of wavelength of 5000 Å being used, has $\mu_0 = 1.553$ and $\mu_e = 1.544$. (5)

[This question paper contains 4 printed pages.]

Sr. No. of Question Paper : 2292 **F-4** **Your Roll No................**

Unique Paper Code : 2351202

Name of the Course : **B.Sc. (Hons.) MATHEMATICS**

Name of the Paper : Differential Equations-I [DC–1]

Semester : II

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **all** questions.
3. Use of scientific calculator is allowed.

SECTION I

1. Attempt any **three** of the following:

(a) Solve the differential equation

$$(3x - y + 1)dx - (6x - 2y - 3)dy = 0.$$

(b) Show that the transformation $v = y^{1-n}$ reduces the Bernoulli equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

to a linear equation in v, where $n \neq 1 = 0$ *or* 1.

(c) Solve the initial value problem

$$(x^2 + y^2)dx - 2xydy = 0, \; y(0) = 2 .$$

(d) Solve the differential equation

$$(2x \cos y + 3x^2y)dx + (x^3 - x^2 \sin y - y)dy = 2.$$ (5+5+5)

2. Attempt any **one** of the following:

(a) Suppose that you discover in your attic an overdue library book on which your grandfather owed a fine of 30 cents 100 years ago. If an overdue fine

P.T.O.

grows exponentially at a 5% annual rate compounded continuously, how much would you have to pay if you returned the book today?

(b) At time $t = 0$ the bottom plug (at the vertex) of a full conical water tank 16 ft high is removed. After 1h the water in the tank is 9 ft deep. When will the tank be empty? (5)

SECTION II

3. Attempt any **two** of the following:

(a) Consider the American system of two lakes: Lake Erie feeding into Lake Ontario. Assume that the volume in each lake remains constant and Lake Erie is the only source of pollution for Lake Ontario. Also suppose that only unpolluted water flows into Lake Erie.

(i) Write down a differential equation system describing the concentration of pollution in each of the lake, define all variables and parameters as required.

(ii) Solve the differential equation system.

(b) Let $N(t)$ be the number of school teachers who have adopted a new technology in their teaching methodology. It is assumed that the rate at which the teachers adopt the technology is proportional to both the number who have adopted the technology and the fraction of the population of the teachers who have not adopted the technology. The process can be modelled as

$$\frac{dN}{dt} = aN\left(1 - \frac{N}{N^*}\right)$$

where N^* is the total population of the teachers.

(i) Solve the above equation to find $N(t)$.

(ii) If $N^* = 10000$ and $a = 0.5$, find how long will it take for 80% of the teachers population to adopt the technology if initially 100 teachers had adopted it.

(iii) Find how long will it take for all the teachers population to adopt the technology.

(c) Tetracycline is an antibiotic prescribed for a range of problem, from acne to acute infections. A course is taken orally and the drug moves from the GI-Tract through the bloodstream, from which it is removed by kidneys and excreted in the urine.

(i) Write down the word equations which describe the movement of a drug through the body, using three compartments: GI- Tract, bloodstream and urinary tract. Note that urinary tract can be considered as an absorbing compartment, that is, the drug enters but is not removed from the urinary tract

(ii) From the word equations develop the differential equation system which describes this process, define all variables and parameters as required. (7½,7½)

SECTION III

4. Attempt any **two** of the following:

(a) Make the substitution $v = \ln x$, find the general solution (for $x > 0$) of the differential equation $x^2y'' + xy' - 9y = 0$

(b) The roots of the characteristic equation of a certain differential equation are $3, -5, 0, 0, 0, 0, -5, 2 \pm 3i$, and $2 \pm 3i$. Find the general solution of this homogeneous differential equation.

(c) Solve the initial value problem:

$3y^{(3)} + 2y'' = 0,\ y(0) = -1,\ y'(0) = 0,\ y''(0) = 1$ (5+5)

5. Attempt any **two** of the following:

(a) Set up the general form of a particular solution y_p of the differential equation

$$(D^2 - 4)(D - 1)^3 y = xe^x + e^{2x} + e^{-2x} \text{ where } D = \frac{d}{dt}$$

using the method of undetermined coefficients.

(b) Use the method of variation of parameters to find a particular solution of the differential equation

$$y'' + 3y' + 2y = 4e^x$$

(c) Use the method of undetermined coefficients to find the particular solution of the differential equation

$y^{(3)} + y'' = x + e^{-x}$ (5+5)

SECTION IV

6. Attempt any **two** of the following:

(a) A model for the spread of a disease, where all infectives recover from the disease and become susceptibles again, is given by the coupled differential equations

$$\frac{dS}{dt} = \beta SI + \alpha I, \quad \frac{dI}{dt} = \beta SI - \alpha I$$

where α and β are positive constants, *S(t)* denotes the number of susceptibles and *I(t)* denotes the number of infectives.

(i) Use the chain rule to find a relation between S and *I*, given the initial numbers of susceptibles and infectives are So and i_0 respectively.

(ii) Draw a sketch of typical phase-plane trajectories. Deduce the directions along the trajectories.

(iii) Using the phase plane diagram describe how the number of infectives changes with time.

(b) A population of sterile rabbits *X(t)* is prayed upon by a population of foxes *Y(t)*. A model of this population interaction is the pair of differential equations

$$\frac{dX}{dt} = -aXY, \quad \frac{dY}{dt} = bXY - cY$$

where *a, b* and *c* are positive constants.

(i) Use the chain rule to find a relationship between the density of foxes and density of rabbits, the initial numbers of rabbits and foxes are x_0 and Y_0 respectively.

(ii) Find and sketch directions of trajectories in the phase plane.

(iii) According to the model is it possible for the foxes to completely wipe out the rabbit population?

(c) The following battle model represents two armies where both the armies used aimed fire, and for one of the armies there is significant loss due to desertion (at a constant rate). The number of soldiers, *R* and *B* , satisfy the coupled differential equations

$$\frac{dR}{dt} = -a_1 B - c, \quad \frac{dB}{dt} = -a_2 R,$$

where a_1, a_2 , and *c* are positive constants.

(i) Use the chain rule to find a relation between *B* and *R*, given the initial number of blue and red soldiers are b_0 *and* r_0 respectively.

(ii) For $a_1 = a_2 = c = 0.01$, sketch typical phase-plane trajectories, giving directions along the trajectories. (10+10)